A New and Very Long Proof of the Pythagoras Theorem
By Way of a Proposition on Isosceles Triangles

by

Kaushik Basu

The World Bank
1818 H Street, NW
Washington, DC 20433
and
Department of Economics
Uris Hall, Cornell University
Ithaca, New York 14853
kb40@cornell.edu

Abstract

This paper provides a new proof of the Pythagoras Theorem on right-angled triangles via two new lemmas pertaining to, respectively, isosceles triangles and right-angled triangles, which are of pedagogical value in themselves.

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1. Introduction

There is an abundance of proofs available for Pythagoras' Theorem on right-angled triangles, from Pythagoras' own alleged proof in the 6th century B.C.[1], through Euclid's proof[2], the proof by Thabit ibn Qurra of Baghdad in the 9th century, the Indian 12th-century mathematician Bhaskara's proof, to the one by the 20th President of the United States James Garfield, who published his paper in 1876, five years before taking up office as President[3].

The aim of this paper is to present a new, and a rather long proof of this theorem. In doing so, I have to face the inevitable question, "Why?" This is especially so because an important principle of mathematics is brevity. Many an original, long proof of a theorem has been cast aside by the subsequent discovery of shorter proofs. The best example of this, within the realm of my own interests, is Kenneth Arrow's celebrated Impossibility Theorem, pertaining to voting systems (Arrow 1951). When it comes to proving it, Arrow's original proof has been superseded almost purely on grounds of its length and the availability now of much shorter proofs, such as those of Amartya Sen (1970) and John Geanakoplos (2005).

How then can one justify presenting a new and longer proof of Pythagoras' theorem? The only way to answer this is to invoke another Greek, Constantine Cavafy and his classic poem, Ithaca, which describes the long journey to Odysseus' home island. When you reach the island, the poet warns the reader, you are likely to be disappointed, for it will have little new to offer. But do not be disappointed, Cavafy tells the reader, for Ithaca's charm is the journey itself.

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[1] The use of the word 'alleged' requires explanation. To the best of our knowledge the theorem was proved fully during the time of Pythagoras. But since Pythagoras also happened to be a prominent philosopher with a cult-like following, it is possible that the theorem was proved by someone else and attributed to him. The cult had many strange beliefs, including a pledge of secrecy and abstinence from eating beans. The cult ultimately broke down; according to Bertrand Russell, because some disciples cheated (and ate beans).
For several different proofs of the Pythagorean theorem see: http://www.cut-the-knot.org/pythagoras/
[2] A commonly used edition is the one published in 1956 and cited fully in the References under Euclid (1956).
[3] President Garfield published his proof in the **New England Journal of Education** in 1976, when he was a Congressman from Ohio. For a discussion of Garfield's proof see Lamb (2012). One of the best illustrations of President Garfield's proof is to be found on the Khan Academy website:
https://www.khanacademy.org/math/basic-geo/basic-geo-pythagorean-topic/basic-geo-pythagorean-proofs/v/garfield-s-proof-of-the-pythagorean-theorem

"Ithaca gave to you the beautiful journey;
without her you'd not have set upon the road.
But she has nothing left to give you any more."

In the present paper, I take reader to the familiar, final theorem via two novel propositions or lemmas, one which pertains to isosceles triangles and the other to right-angled triangles. They establish some interesting properties of isosceles and right-angled triangles. It is hoped that these lemmas, and especially the one on isosceles triangles, will be of interest in themselves[4]. As Cavafy says in the same poem:

"Hope that the road is a long one.
Many may the summer mornings be
when—with what pleasure, with what joy—
you first put in to harbors new to your eyes;"[5]

Section 2 states the main theorem; sections 3 and 4 state and prove the two lemmas and the proof of the main theorem is provided in section 5. The notation used in this paper is section-specific.

2. Pythagoras' Theorem

The theorem that we are setting out to prove, the Pythagoras' Theorem, says the following. Take any right-angled triangle, in which the hypotenuse has a length of c and the other two sides lengths of a and b. The theorem asserts that $c^2 = a^2 + b^2$.

The theorem is now proved via two lemmas, stated and proved in the next two sections.

3. The Isosceles Lemma

Consider an isosceles triangle, in which no angle is greater than 90^0. Such an isosceles triangle is illustrated below, as ABC.

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[4] In defense of this foray into geometry, I may add that while the present paper is an exercise in pure geometry, it is the outcome of a long-standing interest of mine in using geometry to understand economics (Basu, 1992).
[5] The translations from Greek are by Daniel Mendelsohn. The reader interested in the full poem will find it at: http://www.cavafy.com/poems/content.asp?id=259&cat=1 , but is strongly encouraged to read the other poems of Cavafy as well, especially by the same translator (see Cavafy, 2009).

Let the two equal sides be of length c and let the length of CB be r.[6]

Now from the vertex C drop a perpendicular to the side AB and label the touchdown point D, as shown. The length of DB be d.
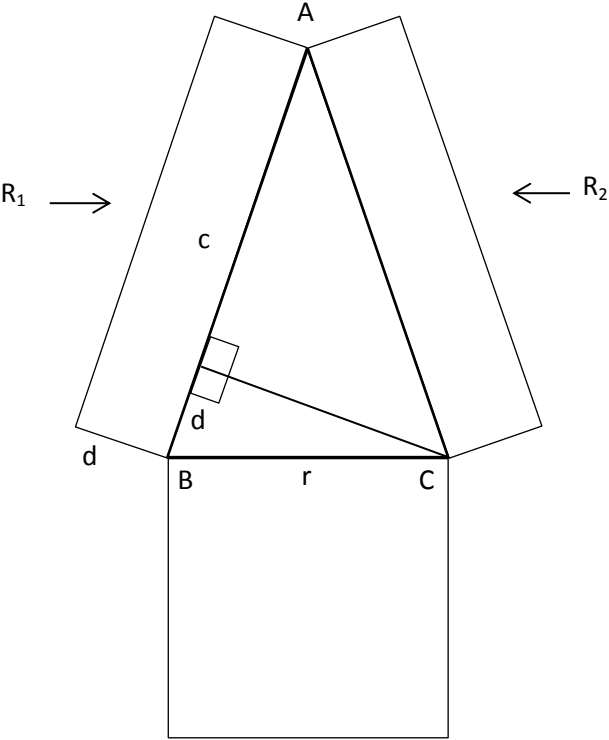
Figure 1
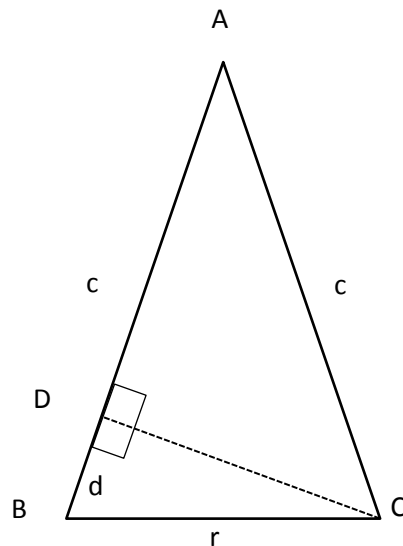
Next draw a rectangle on the side AB, which has height d (and the other side is of course of length c). Draw an identical rectangle on the side AC. These two rectangles are illustrated in Figure 2 and marked R_1 and R_2.

The lemma that I call the Isosceles Lemma and prove below is the following. The sum of the areas of the two rectangles R_1 and R_2 is exactly equal to the area of the square on the third side, BC. In other words, what I am going to prove, is, using the notation introduced in this section, $r^2 = 2dc$

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[6] In case the many "equal to's" that appears in what follows get tiring, I may warn the reader with Nigel Molesworth's observation in **Down with Skool** (see Willans and Searle, 1953): "To do geom you hav to make a lot of things equal to each other when you can see perfectly well that they don't. This agane is due to Pythagoras and it formed much of his conversation at brekfast."

Figure 2

To prove the lemma, take the same triangle as shown in Figures 1 and 2, and drop a perpendicular from A to the side BC. Let the touchdown point be E, as illustrated in Figure 3.

Figure 3
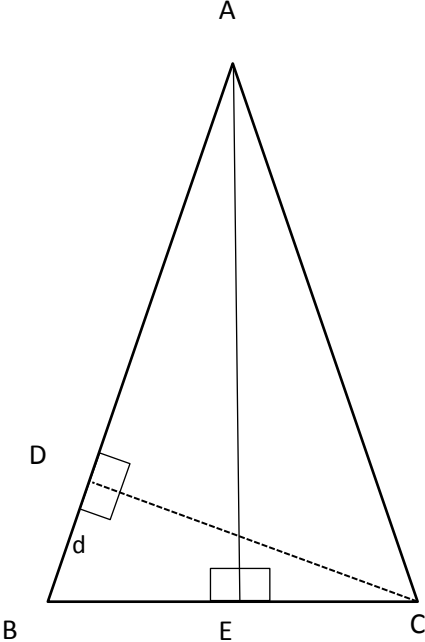
Clearly E bisects BC in two equal parts. Hence, BE = EC = r/2. It is easy to see that triangles AEB and CDB are similar. That is, their only (possible) difference is in size. Hence, the length of DB is to BC is what BE is to AB. Using the notation in this section

$$\frac{d}{r} = \frac{(\frac{r}{2})}{c}$$

Hence, $r^2 = 2dc.$

This completes the proof of the Isosceles Lemma.

Corollary   In the special case in which the isosceles triangle happens to be right-angled, the Isosceles Lemma implies the Pythagoras Theorem.

To see this, think of the angle at vertex A, in Figure 3, to be 90^0. It will be obvious that for such an isosceles triangle, the point D will coincide with A. Hence, in that case d = c. By inserting this in the above equation, we get $r^2 = 2c^2$, which is what the Pythagoras Theorem would assert.

This is of course not enough for what we are setting out to do in this paper, since the corollary applied to only right-angled isosceles triangles. For the full proof of the Pythagorean theorem we have to prove this for all right-angled triangles.

4. The Right-Angled Lemma

In this section I prove what I shall refer to as the "Right-angled Lemma".

Consider a right-angled triangle, such as ABF in Figure 4. From the vertex B drop a line to the side AF such that where it makes contact with AF--call it G--is such that AG = AB. In other words ABG is an isosceles triangle. Let the length of AF be c and FG be f.

Figure 4
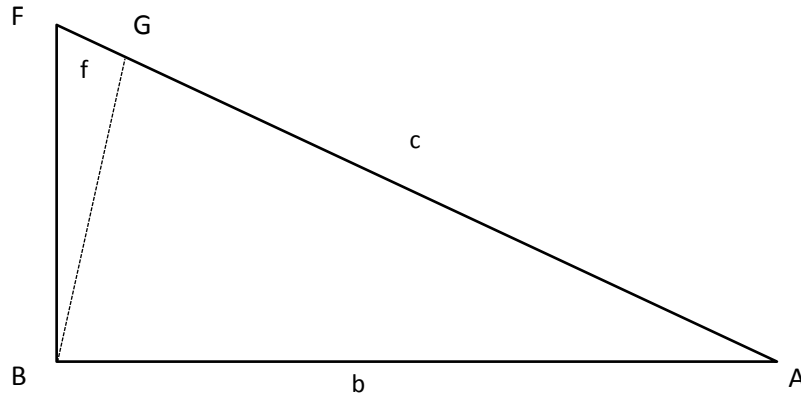
Next draw a rectangle on the side AF and also one on the side AB which have a height of f. These are illustrated in Figure 5 and marked as rectangle R_3 and R_4.

Figure 5
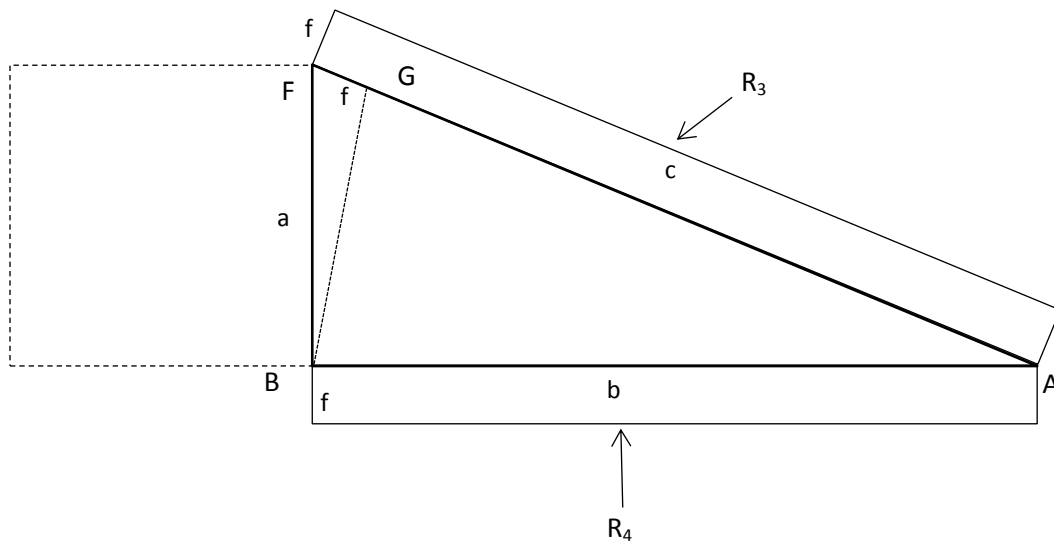
The lemma that I call the "Right-Angled Lemma" and will prove presently says the following. The sum of the areas of the two rectangles R_3 and R_4 is exactly equal to the square on the side FB. In other words, using the notation introduced in this section and referring to the length of AB and BF by b and a, the Lemma claims that $a^2 = bf + cf$.

To prove this, take the same triangle ABF in Figures 5 and from vertex F draw a line parallel to BG and call the point, where it touches the straight line AB, H. This is shown in Figure 6.

Figure 6
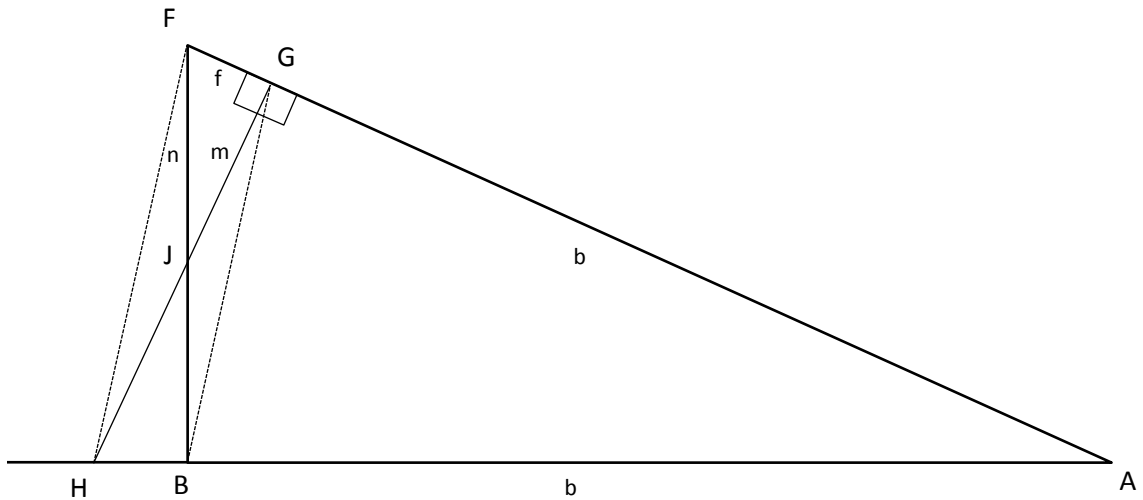
Since FH is parallel to GB and ABG is an isosceles triangle (by construction), AHF must be isosceles as well. Hence, the length of HB is f. Now draw a line linking H and G, and mark the point where HG cuts FB as J. Let FJ = n and JB = m. Clearly, it follows HJ = n and JG = m.

It is easy to see GFJ and BFA are similar triangles. Hence

$$\frac{n}{m} = \frac{c}{b}$$

Since FJ = n and JB = JG = m,   n + m = a.

Hence, $n = a - m = a - \dfrac{bn}{c}$

      or   nc = ac − bn

      or $n = \dfrac{ac}{b+c}$

Again, since GFJ and BFA are similar,

$$\frac{f}{n} = \frac{a}{c}$$

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From the two previous equations, we know
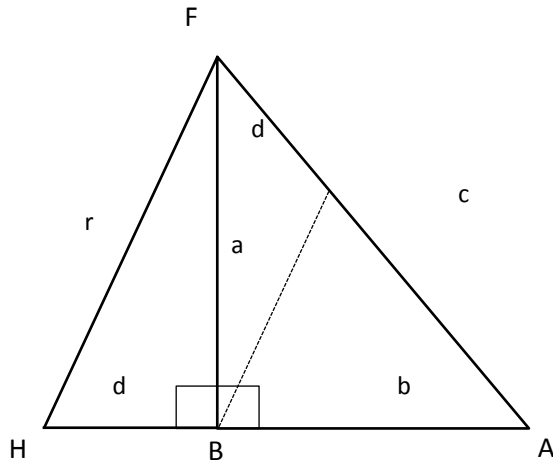
$$\frac{ac}{b+c} = \frac{fc}{a}$$

Therefore, $a^2 = bf + cf.$

5. The Proof of Pythagoras' Theorem

To prove the Pythagoras Theorem, consider a right-angled triangle, BFH, as illustrated in Figure 7. Let the hypotenuse HF be of length r, let BH be of length d, and BF of length a. We have to prove:

$$r^2 = a^2 + d^2.$$

Figure 7

Extend the line HB to the right, up to a point A, such that FA = HA. In other words, AHF is an isosceles triangle. Let the length of AB be b, and the length of AF be c.

By the Isosceles Lemma, we know $r^2 = 2dc$

Or, $r^2 = dc + d(d+b)$

Next, by the Right-angled Lemma, we have

$$a^2 = db + dc$$

The last two above equations imply

$$r^2 = a^2 + d^2.$$

This establishes the Pythagorean Theorem.

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